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Construction of nonnegative symmetric matrices with given spectrum ☆

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Abstract

Let $\sigma = (\lambda_1, \ldots, \lambda_n)$ be the spectrum of a nonnegative symmetric matrix A with the Perron eigenvalue λ_1, a diagonal entry c and let $\tau = (\mu_1, \ldots, \mu_m)$ be the spectrum of a nonnegative symmetric matrix B with the Perron eigenvalue μ_1. We show how to construct a nonnegative symmetric matrix C with the spectrum

$$(\lambda_1 + \max\{0, \mu_1 - c\}, \lambda_2, \ldots, \lambda_n, \mu_2, \ldots, \mu_m).$$

Keywords: Nonnegative inverse eigenvalue problem; Symmetric matrices; Eigenvalues

1. Introduction

The nonnegative inverse eigenvalue problem (NIEP) is the problem of finding necessary and sufficient conditions for a list of complex numbers to be the spectrum of some nonnegative matrix. The symmetric nonnegative eigenvalue problem (SNIEP) can be stated as follows. Find necessary and sufficient conditions for a given list of real numbers to be the spectrum of some nonnegative symmetric matrix.

If there exists a nonnegative matrix A with the spectrum σ, we will say that σ is realizable and that A realizes σ. If A is symmetric we say that σ is symmetrically realizable.

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The problems NIEP and SNIEP have attracted a lot of attention over the years and they both remain unsolved for lists with more than four elements. For a source of literature on the problems we refer the reader to the works and citations that appear in them [1–8].

One of the first general results on NIEP is the following result of Suleĭmanova [9]. That Suleĭmanova's condition is also sufficient for symmetric realizability was proved by Fiedler [10].

Theorem 1. *Let*

$$\lambda_1 > 0 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots \geqslant \lambda_n$$

be real numbers. Then the list

$$(\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_n)$$

is the spectrum of a nonnegative symmetric matrix if and only if

$$\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n \geqslant 0.$$

Results on how to construct new realizable lists from known realizable lists are very useful in solving NIEP and SNEIP. One of the first results of this type was proved for symmetric realizability by Fiedler [10].

Wuwen [11] obtained interesting results for general nonnegative matrices. He proved that if $(\rho, \lambda_2, \ldots, \lambda_n)$ is a realizable list with the Perron eigenvalue ρ and λ_2 is real, then

$$(\rho + t, \lambda_2 \pm t, \lambda_3, \ldots, \lambda_n)$$

is realizable for every $t \geqslant 0$. It is an open question whether this result holds for symmetric realizability.

Laffey [12] generalized Wuwen's result to perturbations of complex eigenvalues. He showed that if $(\rho, a + \mathrm{i}b, a - \mathrm{i}b, \lambda_4, \ldots, \lambda_n)$ is a realizable list with the Perron eigenvalue ρ, then

$$(\rho + 2t, a - t + \mathrm{i}b, a - t - \mathrm{i}b, \lambda_4, \ldots, \lambda_n)$$

is realizable for every $t \geqslant 0$.

Some perturbation results for general nonnegative matrices were presented by Šmigoc in [13] and [14]. In particular, Theorem 11 from [13] is quoted below.

Theorem 2 [13]. *Let $\sigma = (\lambda_1, \ldots, \lambda_n)$ and $\tau = (\mu_1, \ldots, \mu_m)$ be realizable lists with Perron eigenvalues λ_1 and μ_1, respectively, and let σ have a realizing matrix A with a diagonal entry c. Then the list*

$$(\lambda_1 + \max\{0, \mu_1 - c\}, \lambda_2, \ldots, \lambda_n, \mu_2, \ldots, \mu_m)$$

is realizable.

In this work we will show that Theorem 2 holds for symmetric realizablity and give some applications of the result.

2. Diagonal elements of symmetric nonnegative matrices with given spectra

In the literature on the NIEP and SNIEP there has been little consideration of the possible diagonal entries of a matrix realizing a given spectrum. However, this question in the case of real (not necessary nonnegative) symmetric matrices has been the subject of intense analysis and in particular Schur proved his majorization result that the list of eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$,

$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$, of a symmetric matrix A majorizes the list $(a_1, a_2, \ldots, a_n)$ of its diagonal entries. The converse to this result was proved by A. Horn.

Theorem 3. *Let H be a Hermitian matrix. Let $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\alpha = (a_1, a_2, \ldots, a_n)$ be the eigenvalues and the diagonal elements of H, respectively. If they are arranged in the decreasing order*:

$$\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n, \qquad a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n,$$

then

$$\sum_{i=1}^{k} \lambda_i \geqslant \sum_{i=1}^{k} a_i, \tag{1}$$

for all $k = 1, 2, \ldots, n$, with equality for $k = n$.

Conversely, given any two lists of real numbers $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\alpha = (a_1, a_2, \ldots, a_n)$ arranged in the decreasing order that satisfy (1), *there exists a real symmetric matrix H with the eigenvalues σ and the diagonal elements α.*

Extensive analysis and background of this problem can be found in [15].

Here we formulate the following problems for nonnegative and symmetric nonnegative matrices.

Problem 1. Suppose $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a realizable spectrum, determine necessary and sufficient conditions on a list $\alpha = (a_1, a_2, \ldots, a_n)$ such that σ is realized by a matrix A with the diagonal elements α.

Problem 2. Suppose $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a symmetrically realizable spectrum, determine necessary and sufficient conditions on a list $\alpha = (a_1, a_2, \ldots, a_n)$ such that σ is realized by a symmetric matrix A with the diagonal elements α.

Both questions are very difficult and are open even when the realizability of the list σ is very easy to prove. For example, the answer is not known when all the eigenvalues are positive real numbers.

When $n = 2$ both problems are equivalent and are easy to solve. Let $\lambda_1 \geqslant \lambda_2$ and $\lambda_1 + \lambda_2 \geqslant 0$. The list $\sigma = (\lambda_1, \lambda_2)$ is realizable by a nonnegative (symmetric) matrix with diagonal elements $a_1 \geqslant a_2$ if and only if $a_1 = \frac{\lambda_1+\lambda_2}{2} + t$ and $a_2 = \frac{\lambda_1+\lambda_2}{2} - t$ for some $t \in [0, \frac{\lambda_1-|\lambda_2|}{2}]$.

Fiedler [10] presented some results related to Problem 2. In particular, he solved the problem for $n = 3$.

Proposition 4. *Let $\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3$ and $a_1 \geqslant a_2 \geqslant a_3 \geqslant 0$ be given real numbers such that the list $(\lambda_1, \lambda_2, \lambda_3)$ is realizable. Then there exists a nonnegative symmetric matrix with the spectrum $(\lambda_1, \lambda_2, \lambda_3)$ and the diagonal entries (a_1, a_2, a_3) if and only if the following conditions hold*:

(1) $\lambda_1 \geqslant a_1$,
(2) $\lambda_2 \leqslant a_1$,
(3) $\lambda_3 \leqslant a_3$,
(4) $\lambda_1 + \lambda_2 + \lambda_3 = a_1 + a_2 + a_3$.

Fiedler [10] also proved the following necessary conditions.

Theorem 5. *If $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_n$ and $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$ are eigenvalues and diagonal elements of a nonnegative symmetric matrix, then*

$$\sum_{i=1}^{s} \lambda_i + \lambda_k \geqslant \sum_{i=1}^{s-1} a_i + a_{k-1} + a_k$$

for all s and k, $1 \leqslant s < k \leqslant n$.

The following result gives us a way to control the diagonal elements of a realizing matrix as we increase the Perron eigenvalue. Let A be a nonnegative matrix with the spectrum $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, the Perron eigenvalue λ_1 and diagonal elements $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$. Then for every $\epsilon > 0$ and for every $k \in \{1, 2, \ldots, n\}$ there exists a nonnegative matrix B with the spectrum

$$(\lambda_1 + \epsilon, \lambda_2, \ldots, \lambda_n)$$

and diagonal elements $(a_1, \ldots, a_{k-1}, a_k + \epsilon, a_{k+1}, \ldots, a_n)$. For the proof and use of this result see [13].

The same proof does not work for symmetric nonnegative matrices, but we will prove that the result is still true. The following lemma tells us that we can increase a diagonal element of a nonnegative symmetric matrix when we increase the Perron eigenvalue while preserving nonnegativity and symmetry.

Lemma 6. *Suppose that A is an irreducible nonnegative symmetric matrix with a diagonal element a_{11} and the spectrum $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, where λ_1 is the Perron eigenvalue. Then for all $t \geqslant 0$ there exists a nonnegative symmetric matrix B with a diagonal element $b_{11} = a_{11} + t$ and the spectrum $(\lambda_1 + t, \lambda_2, \ldots, \lambda_n)$. The rest of the diagonal elements of the matrix B are the same as the diagonal elements of the matrix A.*

Proof. Write

$$A = \begin{pmatrix} a_{11} & u^{\mathrm{T}} \\ u & A_{22} \end{pmatrix},$$

where u is a nonnegative vector and A_{22} is an $(n-1) \times (n-1)$ nonnegative symmetric matrix.

Let $(d_2, \ldots, d_n)$ be the spectrum of A_{22} and let u_i, $i = 2, \ldots, n$, be orthogonal eigenvectors of A_{22} with $A_{22}u_i = d_i u_i$. Since A is irreducible we have: $|d_i| < \lambda_1$ for $i = 2, \ldots, n$.

We can write:

$$u = \sum_{j=2}^{n} \gamma_j u_j$$

for some real numbers γ_j.

We will prove that we can find a desired matrix of the following form:

$$B = \begin{pmatrix} a_{11} + t & u'^{\mathrm{T}} \\ u' & A_{22} \end{pmatrix}, \tag{2}$$

where $u' = \sum_{i=2}^{n} \gamma_i' u_i$.

Using Theorem 4.3.10 in [16] it is easy to prove the existence of a matrix B of the form (2) with the desired spectrum. Surprisingly, we will be able to show that the vector u' in such a solution can be chosen to be nonnegative.

Let U be the orthogonal matrix with columns u_i. Then

$$\begin{pmatrix} 1 & 0 \\ 0 & U^{\mathrm{T}} \end{pmatrix} A \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} = \begin{pmatrix} a_{11} & \gamma_2 & \cdots & \gamma_n \\ \gamma_2 & d_2 & & \\ \vdots & & \ddots & \\ \gamma_n & & & d_n \end{pmatrix}.$$

Since A has spectrum σ we have:

$$(x-\lambda_1)\prod_{i=2}^{n}(x-\lambda_i) = (x-a_{11})\prod_{i=2}^{n}(x-d_i) - \sum_{i=2}^{n}\gamma_i^2\prod_{j\neq i}(x-d_j). \tag{3}$$

Let $c_2, \ldots, c_m$ be all the distinct eigenvalues of the matrix A_{22} and let the eigenvalue c_i have multiplicity k_i for $i = 2, \ldots, m$. Then $m \leqslant n-1$ and $\sum_{j=1}^{m} k_j = n-1$.

Interlacing inequalities for symmetric matrices, see for example [17], tell us that if $k_i > 1$, then c_i is the eigenvalue of A with multiplicity at least $k_i - 1$. Let $(\mu_1, \mu_2, \ldots, \mu_m)$ be the list obtained from the list $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ by leaving out $k_i - 1$ instances of the eigenvalue c_i for $i = 1, \ldots, m$. In this notation let $\mu_1 = \lambda_1$. For $i = 1, \ldots, m$ we define:

$$\delta_i = \sum_{j, d_j = c_i} \gamma_j^2.$$

Dividing Eq. (3) by $\prod_{j=1}^{m}(x-c_j)^{k_j-1}$ gives us the following equation:

$$(x-\mu_1)\prod_{i=2}^{m}(x-\mu_i) = (x-a_{11})\prod_{i=2}^{m}(x-c_i) - \sum_{i=2}^{m}\delta_i\prod_{j\neq i}(x-c_j). \tag{4}$$

For the matrix B of the form (2) we have:

$$\begin{pmatrix} 1 & 0 \\ 0 & U^{\mathrm{T}} \end{pmatrix} B \begin{pmatrix} 1 & 0 \\ 0 & U \end{pmatrix} = \begin{pmatrix} a_{11}+t & \gamma_2' & \cdots & \gamma_n' \\ \gamma_2' & d_2 & & \\ \vdots & & \ddots & \\ \gamma_n' & & & d_n \end{pmatrix}.$$

We want the matrix B to have the spectrum $(\lambda_1 + t, \lambda_2, \ldots, \lambda_n)$, therefore we want the following equation to hold:

$$(x-\lambda_1-t)\prod_{i=2}^{n}(x-\lambda_i) = (x-a_{11}-t)\prod_{i=2}^{n}(x-d_i) - \sum_{i=2}^{n}(\gamma_i')^2\prod_{j\neq i}(x-d_j). \tag{5}$$

Define

$$\delta_i' = \sum_{j, d_j = c_i} \gamma_j'^2.$$

We divide Eq. (5) by $\prod_{j=1}^{m}(x-c_j)^{k_j-1}$ to obtain:

$$(x-\mu_1-t)\prod_{i=2}^{m}(x-\mu_i) = (x-a_{11}-t)\prod_{i=2}^{m}(x-c_i) - \sum_{i=2}^{m}\delta_i'\prod_{j\neq i}(x-c_j). \tag{6}$$

Now we put $x = c_k$ in (4) and (6) for $k = 2, \ldots, m$. We observe that $\delta_k = 0$ implies $\prod_{i=2}^{m}(c_k - \mu_i) = 0$ and $\delta_k' = 0$. For $\delta_k \neq 0$ we get:

$$\frac{\delta'_k}{\delta_k} = \frac{c_k - \mu_1 - t}{c_k - \mu_1}.$$

Hence:

$$\sum_{j,d_j=c_i} \gamma_j'^2 = \left(\frac{\lambda_1 + t}{\lambda_1}\right)\left(\frac{1 - \frac{c_k}{\lambda_1+t}}{1 - \frac{c_k}{\lambda_1}}\right)\left(\sum_{j,d_j=c_i} \gamma_j^2\right)$$

for $k = 2, \ldots, m$.

The above equation will be satisfied for:

$$\gamma_i' = \sqrt{\frac{\lambda_1 + t}{\lambda_1}}\left(\frac{1 - \frac{d_i}{\lambda_1+t}}{1 - \frac{d_i}{\lambda_1}}\right)^{1/2} \gamma_i.$$

If $\delta_k = 0$, we define $\gamma_i' = 0$ for all i with $d_i = c_k$.

Observe that since $|d_j| < \lambda_1 < \lambda_1 + t$ the series expansions of the matrices $(I - \frac{1}{\lambda_1}A_{22})^{-1/2}$ and $(I - \frac{1}{\lambda_1+t}A_{22})^{1/2}$ converge. We define:

$$W = \left(I - \frac{1}{\lambda_1}A_{22}\right)^{-1/2}\left(I - \frac{1}{\lambda_1 + t}A_{22}\right)^{1/2}$$

and note that:

$$\sqrt{\frac{\lambda_1 + t}{\lambda_1}} W \sum_{j=2}^{n} \gamma_j u_j = \sqrt{\frac{\lambda_1 + t}{\lambda_1}} \sum_{j=2}^{n} \gamma_j \frac{\left(1 - \frac{d_j}{\lambda_1+t}\right)^{1/2}}{\left(1 - \frac{d_j}{\lambda_1}\right)^{1/2}} u_j = \sum_{i=2}^{n} \gamma_i' u_i.$$

Hence can write:

$$u' = \sqrt{\frac{\lambda_1 + t}{\lambda_1}} W u.$$

In the expansion of $(1 - x)^{-1/2}$ all coefficients are positive, while in the expansion of $(1 - x)^{1/2}$ all but the first coefficient are negative:

$$(1 - x)^{1/2} = 1 - \sum_{j=1}^{\infty} g_j x^j,$$

$$(1 - x)^{-1/2} = 1 + \sum_{j=1}^{\infty} f_j x^j,$$

where $g_j \geqslant 0$ and $f_j \geqslant 0$ for $j = 1, 2, \ldots$

Since the coefficient of x^k in the product $(1 - x)^{1/2}(1 - x)^{-1/2}$ is

$$f_k - f_{k-1}g_1 - \cdots - f_1 g_{k-1} - g_k$$

we have:

$$f_k = f_{k-1}g_1 + \cdots + f_1 g_{k-1} + g_k \tag{7}$$

for $k \geqslant 1$.

The coefficient of A_{22}^k in the expansion of W is

$$c_k = \frac{f_k}{\lambda_1^k} - \frac{f_{k-1}g_1}{\lambda_1^{k-1}(\lambda_1+t)} - \cdots - \frac{g_k}{(\lambda_1+t)^k}. \tag{8}$$

Using (7) we can write:

$$c_k = \frac{1}{\lambda_1^k}\left(f_{k-1}g_1\left(1-\frac{\lambda_1}{\lambda_1+t}\right)\right.$$
$$\left. + f_{k-2}g_2\left(1-\left(\frac{\lambda_1}{\lambda_1+t}\right)^2\right) + \cdots + g_k\left(1-\left(\frac{\lambda_1}{\lambda_1+t}\right)^k\right)\right).$$

Since $0 < \lambda_1 < \lambda_1 + t$ the coefficient c_k is positive. Hence the matrix W has nonnegative (in fact, positive) entries. Since W and u are nonnegative and

$$u' = \sqrt{\frac{\lambda_1+t}{\lambda_1}}Wu,$$

u' is nonnegative.

The polynomials

$$(x-\lambda_1-t)\prod_{i=2}^{n}(x-\lambda_i)$$

and

$$(x-a_{11}-t)\prod_{i=2}^{n}(x-d_i) - \sum_{i=2}^{n}(\gamma_i')^2\prod_{j\neq i}(x-d_j)$$

differ by a multiple of $\prod_{j=2}^{n}(x-d_j)$, they are both monic of degree n and their coefficients of x^{n-1} agree. Hence they are equal. This proves our lemma. □

Repeated use of Lemma 6 gives us the following Corollary.

Corollary 7. *Let A be an irreducible nonnegative symmetric matrix with the spectrum $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, the Perron eigenvalue λ_1 and the diagonal elements $(a_1, a_2, \ldots, a_n)$. Let $t > 0$ and let $t_i, i = 1, 2, \ldots, n$, be nonnegative numbers such that $\sum_{i=1}^{n} t_i = t$. Then there exists a nonnegative symmetric matrix B with the spectrum $(\lambda_1 + t, \lambda_2, \ldots, \lambda_n)$ and the diagonal elements*

$$(a_1 + t_1, a_2 + t_2, \ldots, a_n + t_n).$$

3. Symmetric nonnegative inverse eigenvalue problem

In this section we will consider SNIEP. We will present some results that combine realizability and diagonal elements of realizing matrices. Using Lemma 6 we prove our main result.

Theorem 8. *Let A be an $n \times n$ irreducible nonnegative symmetric matrix with the Perron eigenvalue λ_1, the spectrum $(\lambda_1, \ldots, \lambda_n)$ and a diagonal element c. Let B be an $m \times m$ nonnegative symmetric matrix with the Perron eigenvalue μ_1 and the spectrum $(\mu_1, \mu_2, \ldots, \mu_m)$.*

(1) *If $\mu_1 \leqslant c$, then there exists an $(n+m-1) \times (n+m-1)$ nonnegative symmetric matrix C with the spectrum*

$$(\lambda_1, \ldots, \lambda_n, \mu_2, \ldots, \mu_m).$$

(2) *If $c \leqslant \mu_1$, then there exists an $(n+m-1) \times (n+m-1)$ nonnegative symmetric matrix C with the spectrum*

$$(\lambda_1 + \mu_1 - c, \lambda_2, \ldots, \lambda_n, \mu_2, \ldots, \mu_m).$$

Proof. To prove the first item in the theorem we use Lemma 6 for the matrix B to find an $n \times n$ symmetric nonnegative matrix B_0 with spectrum $(c, \mu_2, \ldots, \mu_m)$. Let u be the Perron eigenvector of the matrix $B_0 : B_0 u = cu$ and let the matrix A be of the form:

$$A = \begin{pmatrix} A_1 & a \\ a^{\mathrm{T}} & c \end{pmatrix}.$$

We define:

$$M = \begin{pmatrix} A_1 & au^{\mathrm{T}} \\ ua^{\mathrm{T}} & B_0 \end{pmatrix}.$$

Let U be a real orthogonal matrix with the first column u such that:

$$D = U^{\mathrm{T}} B_0 U = \begin{pmatrix} c & & & \\ & \mu_2 & & \\ & & \ddots & \\ & & & \mu_m \end{pmatrix}$$

and let

$$V = \begin{pmatrix} I_{n-1} & 0 \\ 0 & U \end{pmatrix}.$$

Then

$$\begin{aligned} V^{\mathrm{T}} M V &= \begin{pmatrix} I_{n-1} & 0 \\ 0 & U^{\mathrm{T}} \end{pmatrix} \begin{pmatrix} A_1 & au^{\mathrm{T}} \\ ua^{\mathrm{T}} & B_0 \end{pmatrix} \begin{pmatrix} I_{n-1} & 0 \\ 0 & U \end{pmatrix} \\ &= \begin{pmatrix} A_1 & au^{\mathrm{T}} U \\ U^{\mathrm{T}} ua^{\mathrm{T}} & U^{\mathrm{T}} B_0 U \end{pmatrix} \\ &= \begin{pmatrix} A_1 & a & 0 & \ldots & 0 \\ a^{\mathrm{T}} & c & & & \\ 0 & & \mu_2 & & \\ \vdots & & & \ddots & \\ 0 & & & & \mu_m \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix}. \end{aligned}$$

It follows that the matrix M has the spectrum

$$(\lambda_1, \ldots, \lambda_n, \mu_2, \ldots, \mu_m).$$

The matrix M is clearly nonnegative and symmetric. This proves the first item in the theorem.

Now suppose that $c < \mu_1$. We apply Lemma 6 to the matrix A to find an $n \times n$ nonnegative matrix A_0 with diagonal entry μ_1 and spectrum

$$(\lambda_1 + \mu_1 - c, \lambda_2, \ldots, \lambda_n).$$

The first item of the theorem tells us that there exists a nonnegative symmetric matrix C with eigenvalues: $(\lambda_1 + \mu_1 - c, \lambda_2, \ldots, \lambda_n, \mu_2, \ldots, \mu_m)$. $\square$

Using this result, we can constructively obtain many symmetrically realizable lists. For example, there has been a lot of work done on the constructive methods in SNIEP [18–20]. Theorem 8 gives us an alternative way to prove these results by joining together 2×2 nonnegative symmetric matrices.

The concept of extreme spectrum and extreme symmetric spectrum was introduced in [21] by Laffey and it has proved useful in solving NIEP and SNIEP. We say that $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is an extreme (symmetric) spectrum if it is (symmetrically) realizable, but for all $t > 0$, $(\lambda_1 - t, \lambda_2 - t, \ldots, \lambda_n - t)$ is not (symmetrically) realizable.

The list $\sigma = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ is a Perron extreme (symmetric) spectrum if it is (symmetrically) realizable, but for all $t > 0$, $(\lambda_1 - t, \lambda_2, \ldots, \lambda_n)$ is not (symmetrically) realizable.

For example, realizable lists with trace zero are both extreme and Perron extreme.

We combine the concept of extreme and of Perron extreme spectrum in the following question.

Problem 3. Let $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be the spectrum of a nonnegative symmetric matrix and let $t \geqslant 0$. Find the smallest ρ for which $(\rho, \lambda_2 - t, \lambda_3 - t, \ldots, \lambda_n - t)$ is realizable.

The following theorem gives a bound on the Perron eigenvalue ρ in the above problem that depends on the diagonal elements of the realizing matrix.

Theorem 9. *Let $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be the spectrum of a nonnegative symmetric matrix A with diagonal elements $(a_1, a_2, \ldots, a_n)$, $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$, and let*

$$f(t) = \begin{cases} 0 & \text{for } t \leqslant a_n, \\ (n-k-1)t - \sum_{i=k+1}^{n} a_i & \text{for } a_k \geqslant t > a_{k+1}, \\ (n-1)t - \sum_{i=1}^{n} a_i & \text{for } t \geqslant a_1. \end{cases}$$

Then the list $(\lambda_1 + f(t), \lambda_2 - t, \lambda_3 - t, \ldots, \lambda_n - t)$ is the spectrum of a nonnegative symmetric matrix.

Proof. For $t \leqslant a_n$ the matrix $A - tI$ is nonnegative and has the desired spectrum. Now assume that $a_k \geqslant t > a_{k+1}$ for some $k = 1, \ldots, n-1$. Using Lemma 6 we can obtain a nonnegative symmetric matrix B with diagonal elements $(a_1, a_2, \ldots, a_k, t, \ldots, t)$ and spectrum

$$\left(\lambda_1 + (n-k)t - \sum_{i=k+1}^{n} a_i, \lambda_2, \ldots, \lambda_n\right).$$

The matrix $B - tI$ is nonnegative and has the desired spectrum. $\square$

In particular, in view of the following corollary this result gives a complete answer to the Problem 3 in the case when $t \geqslant a_1$.

Corollary 10. *Let $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ be the spectrum of a nonnegative symmetric matrix A with diagonal elements $(a_1, a_2, \ldots, a_n)$, $a_1 \geqslant a_2 \geqslant \cdots \geqslant a_n$ and let $t \geqslant a_1$. Then the list $(\lambda_1 + (n-1)t - \sum_{i=1}^{n} \lambda_i, \lambda_2 - t, \lambda_3 - t, \ldots, \lambda_n - t)$ is the spectrum of a nonnegative symmetric matrix with trace zero.*

Similar results can be obtained for the spectra of nonnegative but not necessarily symmetric matrices.

4. Examples

In this section we will present some examples of how to find realizable lists using the previous results in this paper.

The solution to Problem 2 for realizable lists with all but the dominant eigenvalue less than or equal to zero was found by Fiedler [10]. In the next example we give an alternative proof of this result.

Example 11 [10]. Let

$$\lambda_1 > 0 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \cdots \geqslant \lambda_n$$

be real numbers, such that

$$\lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n \geqslant 0.$$

Then the list $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ can be realized by a nonnegative symmetric matrix with diagonal elements $(a_1, a_2, \ldots, a_n)$, $a_i \geqslant 0$ for $i = 1, \ldots, n$, if and only if

$$a_1 + a_2 + \cdots + a_n = \lambda_1 + \lambda_2 + \lambda_3 + \cdots + \lambda_n.$$

To prove this statement we first observe that the list

$$\sigma = (-\lambda_2 - \cdots - \lambda_n, \lambda_2, \ldots, \lambda_n)$$

is realizable by Theorem 1. Since σ has trace zero, every nonnegative matrix with spectrum σ has all its diagonal elements equal to zero. Now we use Corollary 7 to finish the proof.

Next, we give a numerical example. We will present a list of real numbers that is not symmetrically realizable, but which becomes symmetrically realizable if we add a zero to it. Note that this cannot occur for lists with four elements.

Example 12. In [22] it was shown that $\lambda_1 + \lambda_3 + \lambda_4 \geqslant 0$ is a necessary condition for the list of real numbers

$$\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \lambda_4 \geqslant \lambda_5$$

to be the spectrum of some nonnegative symmetric matrix.

This implies that the list $(7, 5, -4, -4, -4)$ is not realizable by a nonnegative symmetric matrix. We will show that the list $\sigma = (7, 5, 0, -4, -4, -4)$ is the spectrum of a nonnegative symmetric matrix.

First we show that the list $(7, 5, 0, -4)$ can be realized by a nonnegative symmetric matrix with diagonal elements $(4, 4, 0, 0)$.

The realizing matrix will be of the form

$$A = \begin{pmatrix} 4 & 0 & b & 0 \\ 0 & 4 & 0 & e \\ b & 0 & 0 & g \\ 0 & e & g & 0 \end{pmatrix}.$$

By comparing the characteristic polynomial of the matrix A with the polynomial $(x-7)(x-5)(x+4)x$, we see that A will have the spectrum σ if the following equations hold:

$$g = \sqrt{6}, \quad b^2 + e^2 = 23 \quad \text{and} \quad be = 4\sqrt{6}.$$

Even though the above equations do not have a nice solution for b and e it is not difficult to see that they have a positive solution.

Clearly the list $(4, -4)$ is realizable. Now we use Theorem 8. First we obtain a nonnegative symmetric matrix A_1 with the spectrum $(7, 5, 0, -4, -4)$ and a diagonal element 4, then we use Theorem 8 for lists $(7, 5, 0, -4, -4)$ and $(4, -4)$ to obtain a nonnegative symmetric matrix with the spectrum σ.

Next we consider a more general example.

Example 13. The smallest t for which $(3+t, 3-t, -2, -2, -2)$ is the spectrum of a symmetric nonnegative matrix is $t = 1$. This was shown by Loewy and the proof can be found in [23]. Here we consider the question for which t is $(3+t, 3-t, 0, -2, -2, -2)$ the spectrum of a symmetric nonnegative matrix. We use a similar approach to that in the previous example. The matrix

$$\begin{pmatrix} 2 & 0 & 2\sqrt{\frac{2}{3}} & 0 \\ 0 & 2 & 0 & 2\sqrt{\frac{2}{3}} \\ 2\sqrt{\frac{2}{3}} & 0 & 0 & \frac{4}{3} \\ 0 & 2\sqrt{\frac{2}{3}} & \frac{4}{3} & 0 \end{pmatrix}$$

has spectrum $(\frac{10}{3}, \frac{8}{3}, 0, -2)$. We use Theorem 8 to join lists $(\frac{10}{3}, \frac{8}{3}, 0, -2)$ and $(2, -2)$. Now we know that the list $(\frac{10}{3}, \frac{8}{3}, 0, -2, -2)$ is realizable by a nonnegative symmetric matrix with a diagonal element 2. Joining lists $(\frac{10}{3}, \frac{8}{3}, 0, -2, -2)$ and $(2, -2)$ proves that $(3+t, 3-t, 0, -2, -2, -2)$ is realizable with $t = \frac{1}{3}$. We do not know if $t = \frac{1}{3}$ is the smallest possible.

Using Theorem 8 we can find a large set of symmetrically realizable lists. However, this method does not give us all symmetrically realizable lists. We give an example of a symmetrically realizable list that cannot be obtained by using Theorem 8 to join lists of smaller size.

Example 14. The list

$$\sigma = \left(2, \frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, -\frac{1+\sqrt{5}}{2}, -\frac{1+\sqrt{5}}{2}\right)$$

is the spectrum of the matrix

$$A = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Knudsen and McDonald [24] showed that σ a realizable spectrum.

It is easy to see that it is impossible to obtain σ by joining two lists with three elements. If we could obtain σ by joining a list with four elements and a list with two elements, then the list $(2, \frac{-1+\sqrt{5}}{2}, \frac{-1+\sqrt{5}}{2}, -\frac{1+\sqrt{5}}{2})$ would have to be realizable by a nonnegative symmetric matrix with diagonal elements $(\frac{1+\sqrt{5}}{2}, 0, 0, 0)$. This contradicts necessary conditions in Theorem 5 for $s = 2$ and $k = 4$.

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